if A is a vector then A·r = scalar, indep of rotation (5)

metric: **for practice**

$$\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \qquad \begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \qquad \begin{pmatrix}-1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}$$

$$\begin{aligned} s &= A^t r \\ &= A^t \Lambda^{-1}\Lambda r \\ &= A^t \Lambda^{-1} \bar{r} \\ &= A^t \Lambda^t \bar{r} \\ &= (\Lambda A)^t \bar{r} \\ &= \bar{A}^t \bar{r} \end{aligned}$$

if $\bar{A} = \Lambda A$

so, if A transforms this way, then it's a vector

$$\begin{aligned} S &= A_\mu r_\mu \\ &= A_\mu \underbrace{\Lambda^{-1}_{\mu\nu}\Lambda_{\nu\alpha}}_{\delta_{\mu\alpha}} r_\alpha \\ &= A_\mu \Lambda^{-1}_{\mu\nu} \bar{r}_\nu \\ &= A_\mu \Lambda_{\nu\mu} \bar{r}_\nu \\ &= \Lambda_{\nu\mu} A_\mu \bar{r}_\nu \\ &= \bar{A}_\nu \bar{r}_\nu \end{aligned}$$

if $\bar{A}_\nu = \Lambda_{\nu\mu} A_\mu$

$$\begin{aligned} S &= A_\mu r^\mu \\ &= A_\mu \underbrace{\Lambda^{-1\mu}{}_\nu \Lambda^\nu{}_\alpha}_{\delta^\mu_\alpha} r^\alpha \\ &= A_\mu \Lambda^{-1\mu}{}_\nu \bar{r}^\nu \\ &= A_\mu \Lambda_\nu{}^\mu \bar{r}^\nu \\ &= \Lambda_\nu{}^\mu A_\mu \bar{r}^\nu \\ &= \bar{A}_\nu \bar{r}^\nu \end{aligned}$$

if $\bar{A}_\nu = \Lambda_\nu{}^\mu A_\mu$

or $\bar{A}^\nu = \Lambda^\nu{}_\mu A^\mu$

consider two 4-vectors $\bar{A}_\mu \bar{B}^\mu = \Lambda_\mu{}^\nu A_\nu \Lambda^\mu{}_\alpha B^\alpha = \Lambda_\mu{}^\nu \Lambda^\mu{}_\alpha A_\nu B^\alpha$
$= \delta^\nu_\alpha A_\nu B^\alpha = A_\nu B^\nu$ ← Lorentz invariant

consider $d\bar{r}^\mu = \Lambda^\mu{}_\nu dr^\nu \Rightarrow$ Lorentz vector

$dr^\nu dr_\nu = -c^2dt^2 + d\vec{x}^2$ = Lorentz invariant

in frame of object $= -c^2 d\tau^2 \Rightarrow d\tau$ is Lorentz invariant

advanced but neat

what about $\dfrac{\partial}{\partial \bar{r}_\mu} = \dfrac{\partial}{\partial r_\nu}\dfrac{\partial r_\nu}{\partial \bar{r}_\mu}$ but: $\bar{r}_\alpha = \Lambda_\alpha{}^\nu r_\nu$

$= \dfrac{\partial}{\partial r_\nu} \underbrace{\dfrac{\partial(\Lambda^{-1}{}_\nu{}^\alpha \bar{r}_\alpha)}{\partial \bar{r}_\mu}}_{=0 \text{ unless } \alpha=\mu}$ $\qquad \Lambda^{-1}{}_\beta{}^\alpha \bar{r}_\alpha = \underbrace{\Lambda^{-1}{}_\beta{}^\alpha \Lambda_\alpha{}^\nu}_{\delta^\nu_\beta} r_\nu$

$\Lambda^{-1}{}_\nu{}^\alpha \bar{r}_\alpha = r_\nu$

$= \Lambda^{-1}{}_\nu{}^\mu \dfrac{\partial}{\partial r_\nu}$

$= \Lambda^\mu{}_\nu \dfrac{\partial}{\partial r_\nu}$ transforms like contravariant

$\partial^\mu \equiv \dfrac{\partial}{\partial r_\mu} = \left(\dfrac{-\partial}{c\partial t} \quad \vec{\nabla}\right)$ $\qquad \partial^\mu \partial_\mu = \square = \underbrace{-\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} + \vec{\nabla}^2}_{\text{wave eqn.}}$

$\partial_\mu = \dfrac{\partial}{\partial r^\mu} = \left(\dfrac{1}{c}\dfrac{\partial}{\partial t} \quad \vec{\nabla}\right)$

Try this approach out:

consider "velocity"

$$\vec{u} = \frac{d\vec{\ell}}{dt} \qquad \vec{\eta} = \frac{d\vec{\ell}}{d\tau} = \gamma\vec{u}$$

"proper velocity"

note: $\eta^0 = \frac{dct}{d\tau} = \gamma c$

how do these transform? done earlier

$$\underbrace{\begin{aligned}\bar{\eta}^\mu &= \Lambda^\mu{}_\nu \eta^\nu \\ \bar{\eta}_\mu &= \Lambda_\mu{}^\nu \eta_\nu\end{aligned}}_{\text{simple since } d\tau \text{ is L.I.}}$$

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momentum: recall $\vec{p} \neq m\vec{u}$ but $\vec{p} = \gamma m\vec{u} = m\vec{\eta}$

what about the "0th" component?

$$p^0 = m\eta^0 = m\frac{d(ct)}{d\tau} = \gamma mc = \frac{\gamma mc^2}{c} = \frac{E}{c}$$

so: $p^\mu p_\mu = -(p^0)^2 + (\vec{p}\cdot\vec{p}) = $ Lorentz Invariant

$= -m^2c^2$ (in rest frame of particle since $\gamma = 1$, $\vec{p} = 0$)

-or- $-\frac{E^2}{c^2} + p^2 = -m^2c^2 \Rightarrow E^2 = p^2c^2 + m^2c^4$

(note $\eta^\mu \eta_\mu = -c^2$ always!)

Work Energy theorem again:

$$\frac{d}{dt}\left(p^{\mu} p_{\mu} = \frac{-E^2}{c^2} + \vec{p}^{\,2} = -m^2c^2 \right)$$

$$\frac{-2E}{c^2}\frac{dE}{dt} + 2\vec{p}\cdot\frac{d\vec{p}}{dt} = 0$$

but $E = \gamma mc^2$, $\vec{p} = \gamma m\vec{u}$, $\frac{d\vec{p}}{dt} = \vec{F}$

$$\frac{-2\gamma mc^2}{c^2}\frac{dE}{dt} + 2\gamma m\,\vec{F}\cdot\vec{u} = 0$$

$$\frac{dE}{dt} = \vec{F}\cdot\vec{u}$$

FORCE $\left(\frac{d\vec{p}}{dt}\right)$ consider boost along x axis

$$\begin{pmatrix}\bar{p}^0\\ \bar{p}^1\\ \bar{p}^2\\ \bar{p}^3\end{pmatrix} = \begin{pmatrix}\gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}p^0\\ p^1\\ p^2\\ p^3\end{pmatrix}$$

$$\begin{pmatrix}c\bar{t}\\ \bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}\gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}ct\\ x\\ y\\ z\end{pmatrix}$$

ie $d\bar{t} = \gamma dt - \frac{\beta\gamma\, dx}{c}$

$$\bar{p}^1 = \frac{-\beta\gamma E}{c} + \gamma p^1$$

so $$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{dp_y}{\gamma dt - \frac{\beta\gamma}{c}dx}$$

$$= \frac{dp_y/dt}{\gamma - \frac{\beta\gamma}{c}\frac{dx}{dt}} = \frac{F_y}{\gamma\left(1 - \frac{\beta}{c}u_x\right)} \qquad \left(u_x = \frac{dx}{dt}\right)$$

$$\bar{F}_z = \frac{F_z}{\gamma\left(1 - \frac{\beta}{c}u_x\right)}$$

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{F_x - \frac{\beta}{c}\left(\frac{dE}{dt}\right)}{\left(1 - \frac{\beta u_x}{c}\right)} = \frac{F_x - \beta\left(\vec{u}\cdot\vec{F}\right)/c}{\left(1 - \beta u_x/c\right)}$$

<u>YUK</u>, unless $\vec{u} = 0$ then $\bar{F}_\perp = \frac{1}{\gamma}F_\perp$; $\bar{F}_\parallel = F_\parallel$

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note: $K^\mu = \frac{dp^\mu}{d\tau}$ "Lorentz Force" transforms <u>much</u> more nicely $\left(K^0 = \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau}\right)$

$$\bar{K}^\mu = \Lambda^\mu{}_\nu K^\nu \Rightarrow \begin{aligned}\bar{K}^0 &= \gamma K^0 - \beta\gamma K^1\\ \bar{K}^1 &= -\beta\gamma K^0 + \gamma K^1\\ \bar{K}^2 &= K^2\\ \bar{K}^3 &= K^3\end{aligned}$$