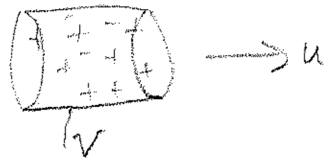
1.

$$\underbrace{\rho = \frac{Q}{V} \qquad \vec{J} = \rho \vec{u}}$$



in frame of charge $\rho_0 = \frac{Q}{V_0}$ but $V = \frac{V_0}{\gamma}$ (by length cont.)

so $\rho = \gamma \rho_0$ $\qquad \vec{J} = \gamma \rho_0 \vec{u}$ $\qquad$ recall $\eta^0 = c\gamma$

$= \frac{\rho_0}{c} \eta^0$ $\qquad = \rho_0 \vec{\eta}$

$c\rho = \rho_0 \eta^0$

$$J^\mu = \rho_0 \eta^\mu = \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix}$$

note $\partial_\mu = \frac{\partial}{\partial x^\mu}$

consider the Lorentz invariant: $\partial_\mu J^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t} \; \frac{\partial}{\partial x} \; \frac{\partial}{\partial y} \; \frac{\partial}{\partial z}\right) \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix}$

$$\boxed{\begin{aligned} &\text{note: } \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \equiv 0 \\ &\mu_0 \vec{\nabla} \cdot \vec{J} + \mu_0 \epsilon_0 \frac{\partial (\vec{\nabla} \cdot \vec{E})}{\partial t} = 0 \qquad \vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \\ &\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \end{aligned}} \longrightarrow \; = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

by continuity

NOTE

next 7 pages are not for 3504, but included for completeness

$\therefore$ without proof: (shown in text)

$$\boxed{\begin{array}{l} \partial_\nu F^{\mu\nu} = \mu_0 J^\mu \\ \\ \partial_\nu G^{\mu\nu} = 0 \end{array}}$$

Maxwell's Eqns

$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

$\Rightarrow \vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

recall: $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$ "vector potential"

so $\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = -\vec{\nabla}\times\frac{\partial\vec{A}}{\partial t}$

so $\vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right) = 0 \Rightarrow \vec{E}+\frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi$

so $\vec{E} = -\left(\vec{\nabla}\phi + \frac{\partial\vec{A}}{\partial t}\right)$

recall $\left.\begin{array}{l}\vec{A}' = \vec{A} + \vec{\nabla}f \\ \phi' = \phi - \frac{\partial f}{\partial t}\end{array}\right\}$ gauge transformations

ie: $\vec{B}' = \vec{\nabla}\times(\vec{A}+\nabla f) = \vec{\nabla}\times\vec{A} + \cancelto{0}{\vec{\nabla}\times\nabla f} = \vec{B}$

$\vec{E}' = -\left(\vec{\nabla}\left(\phi - \frac{\partial f}{\partial t}\right) + \frac{\partial}{\partial t}(\vec{A}+\vec{\nabla}f)\right)$

$= -\left(\vec{\nabla}\phi + \frac{\partial\vec{A}}{\partial t}\right) = \vec{E}$

pick "Lorentz" gauge

$\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}$

so wave eqns become:

(show with things like $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})\dots$)

$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$

$\nabla^2\phi/c - \frac{1}{c^2}\frac{\partial^2\phi/c}{\partial t^2} = -\mu_0 c\rho$

but this is just

$\underbrace{\partial_\mu\partial^\mu}_{\text{scalar}} A^\nu = -\mu_0 \underbrace{J^\nu}_{\text{4-vector}}$

$\Rightarrow A^\nu = \begin{pmatrix}\phi/c \\ A_x \\ A_y \\ A_z\end{pmatrix}$ is a 4-vector

(transforming A^ν can get E & B in diff frames)

∴ without proof
(done in book)

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad \leftarrow \text{notice anti-symmetric}$$

$$= \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$

---

recap:

4-vectors we've seen

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad dx^\mu = \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix} \quad \eta^\mu = \begin{pmatrix} \gamma c \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix} \quad p^\mu = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} \quad J^\mu = \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix}$$

note $p^\mu = m\eta^\mu$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \quad A^\mu = \begin{pmatrix} \phi/c \\ A_x \\ A_y \\ A_z \end{pmatrix}$$

Lorentz scalars we've seen (same in all frames)

$$x^\mu x_\mu = -c^2t^2 + x^2 + y^2 + z^2 = -c^2t^2 + \vec{x}^2$$

$$dx^\mu dx_\mu = -c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2dt^2 + d\vec{x}^2 = -c^2d\tau^2 \quad \Rightarrow d\tau \text{ is L.S.}$$

$$\eta^\mu \eta_\mu = -c^2$$

$$p^\mu p_\mu = -\frac{E^2}{c^2} + \vec{p}^2 = -m^2c^2 \quad \Rightarrow E^2 = \vec{p}^2c^2 + m^2c^4$$

$$\partial^\mu \partial_\mu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2 \equiv \Box^2$$

$$\partial_\mu J^\mu = 0$$

$$\partial_\mu A^\mu = 0 \quad \Rightarrow \text{Lorentz gauge}$$

contra to co: $x_\mu = g_{\mu\nu} x^\nu$

Forces: $\vec{F} = \frac{d\vec{p}}{dt} \quad \leftarrow$ 3 vector

$K^\mu = \frac{dp^\mu}{d\tau} \quad \leftarrow$ 4 vector

E + M: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$

"Electro-Magnetic Field Tensor"

Maxwell's Eqns: $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$

$\partial_\nu G^{\mu\nu} = 0$

(Minkowski) Lorentz Force: $K^\mu = q F^{\mu\nu} \eta_\nu$

Wave Eqn: $\partial_\mu \partial^\mu A^\nu = -\mu_0 J^\nu$ (in Lorentz gauge $\partial_\mu A^\mu = 0$)

$\Box^2 A^\nu = -\mu_0 J^\nu$

Switching frames

$\bar{X}^\mu = \Lambda^\mu{}_\nu X^\nu$ X^ν an arbitrary 4-vector

$\bar{X}_\mu = \Lambda_\mu{}^\nu X_\nu$

ie: $\bar{K}^\mu = \Lambda^\mu{}_\nu K^\nu$

$\bar{\eta}^\mu = \Lambda^\mu{}_\nu \eta^\nu$

Special cases ie dt instead of $d\tau$

normal forces $= \frac{d\vec{p}}{dt}$

ie boost in x

$$\bar{F}_y = \frac{F_y}{\gamma(1 - \beta\frac{u_x}{c})}$$

$$\bar{F}_z = \frac{F_z}{\gamma(1 - \beta\frac{u_x}{c})}$$

$$\bar{F}_x = \frac{F_x - \beta(\vec{u}\cdot\vec{F})/c}{(1 - \beta\frac{u_x}{c})}$$

normal velocities $= \frac{dx}{dt}$

ie boost in x

$$\bar{u}_x = \frac{u_x - v}{(1 - vu_x/c^2)}$$

$$\bar{u}_y = \frac{u_y}{\gamma(1 - vu_x/c^2)}$$

$$\bar{u}_z = \frac{u_z}{\gamma(1 - vu_x/c^2)}$$

$$\bar{F}^{\alpha\beta} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu F^{\mu\nu} = \Lambda^\alpha{}_\mu F^{\mu\nu} (\Lambda^t)_\nu{}^\beta = \underbrace{\Lambda F \Lambda^t}_{\text{as matrices}}$$

ie boost in x

$$\bar{E}_x = E_x \qquad \bar{B}_x = B_x$$
$$\bar{E}_y = \gamma(E_y - vB_z) \qquad \bar{B}_y = \gamma\left(B_y + \frac{vE_z}{c^2}\right)$$
$$\bar{E}_z = \gamma(E_z + vB_y) \qquad \bar{B}_z = \gamma\left(B_z - \frac{vE_y}{c^2}\right)$$

$$\bar{F} = \begin{pmatrix} & & \\ & \Lambda & \\ & & \end{pmatrix}\begin{pmatrix} & & \\ & F & \\ & & \end{pmatrix}\begin{pmatrix} & & \\ & \Lambda^t & \\ & & \end{pmatrix}$$

Λ options: rotate about x: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{pmatrix}$ $s \equiv \sin\theta$, $c \equiv \cos\theta$

y: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & -s \\ 0 & 0 & 1 & 0 \\ 0 & s & 0 & c \end{pmatrix}$

orthogonality $\Lambda^\alpha{}_\beta \Lambda^\gamma{}_\alpha = \delta^\gamma_\beta$

z: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

boost in x: $\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

note: under boosts components of $\vec{E}$ + $\vec{B}$ get mixed up

y: $\begin{pmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

z: $\begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$

In fact, it probably confuses more than enlightens since in a matrix it is hard to tell if the indices are supposed to be superscript or subscript. In fact for anything above rank 2 tensor, it's not possible to write in "matrix form".

Correction to H.O. : 1) in Mathematica it should read

transform = lamda . f . Transpose[lamda]

(makes no difference for boosts, but does affect rotations)

2) on last page:

$$\vec{F} = (\Lambda)(F)(\Lambda^t)$$

---

note: $\partial_\mu J^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}\ \frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right)\begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$ (by continuity)

As Aaron noted in class: note: + since $\partial_\mu = \frac{\partial}{\partial x^\mu}$

---

consider $\eta^\mu \eta_\mu = (c\gamma, \eta_1, \eta_2, \eta_3)\begin{pmatrix} -c\gamma \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = -c^2\gamma^2 + \vec{\eta}^2 = -c^2$ in frame of particle

we use $g = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ but more common to use $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$\Rightarrow \eta^\mu \eta_\mu = (c\gamma\ \eta_1\ \eta_2\ \eta_3)\begin{pmatrix} c\gamma \\ -\eta_1 \\ -\eta_2 \\ -\eta_3 \end{pmatrix} = c^2\gamma^2 - \vec{\eta}^2 = c^2$ in frame of particle

does this make sense? In frame of particle, it moves forward in TIME (η^0) at c. In another frame as $\vec{\eta} \to$ large $\gamma \to$ large such that $c^2\gamma^2 - \vec{\eta}^2 = \text{constant} = c^2$ (L.I.)

A particle moving past you w/ velocity $\vec{u}$ also has its clock running slow, but $dt = \gamma d\tau$ so for you, an increase in the spatial part of the 4-velocity $\vec{\eta} = \frac{d\vec{x}}{d\tau}$ also gives an increase in the time component $\eta^0 = \frac{dct}{d\tau} = c\gamma$

∴ !!ADVANCED!! where else have you seen "similarity" t-forms?

NOT 350H

recall : $\overleftrightarrow{I}\,\vec{\omega}_k = \lambda_k \vec{\omega}_k \Rightarrow \vec{\omega}_k, \lambda_k$

(from lin. alg. or mechanics)

form $\overleftrightarrow{A} = \begin{pmatrix} \hat{\omega}_1 & \cdots \\ \hat{\omega}_2 & \cdots \\ \hat{\omega}_3 & \cdots \end{pmatrix}$ notice: normalized eigenvectors

then $\overleftrightarrow{A}^{-1} = \overleftrightarrow{A}^t$

so $\underbrace{\overleftrightarrow{A}\,\overleftrightarrow{I}\,\overleftrightarrow{A}^t}_{\overleftrightarrow{I}'}\underbrace{\overleftrightarrow{A}\,\vec{\omega}_k}_{\vec{\eta}_k} = \lambda \overleftrightarrow{A}\,\vec{\omega}_k$

$\overleftrightarrow{I}' \quad \vec{\eta}_k = \lambda_k \vec{\eta}_k$

$\overleftrightarrow{I}' = \overleftrightarrow{A}\,\overleftrightarrow{I}\,\overleftrightarrow{A}^t$

$\vec{\eta} = \overleftrightarrow{A}\,\vec{x}$

$\vec{x} = \overleftrightarrow{A}^{-1}\vec{\eta}\ \left(= \overleftrightarrow{A}^t \vec{\eta}\right)$

check: $\hat{x}_1 = \begin{pmatrix} \hat{\omega}_1 & \hat{\omega}_2 & \hat{\omega}_3 \\ \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{\omega}_{1x} \\ \hat{\omega}_{1y} \\ \hat{\omega}_{1z} \end{pmatrix}$ ✓

so we have $\overleftrightarrow{I}' = \overleftrightarrow{A}\,\overleftrightarrow{I}\,\overleftrightarrow{A}^t$ (since $\overleftrightarrow{A}$ is ortho<u>normal</u>)

note: $(\;—\;)(\;|\;)(\;\vdots\;)$

ie: order in which multiplication is carried out doesn't matter

ie $(AB)C = A(BC)$

-vs-

do this so that rows + columns are next to each other + can write like matrix mult.

$$\bar{F}^{\alpha\beta} = \Lambda^\alpha{}_\mu F^{\mu\nu} \Lambda^{\dagger}{}_\nu{}^\beta = \Lambda^\alpha{}_\mu F^{\mu\nu} \Lambda^{t}{}_\nu{}^\beta = (\Lambda)(F)(\Lambda^t)$$

$$= \Lambda^\alpha{}_\mu F^{\mu\nu} \Lambda^\beta{}_\nu$$

$$= \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu F^{\mu\nu}$$

ie: boost in x: $\bar{F} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}(F)\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

rotate about x: $\bar{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{pmatrix}(F)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{pmatrix}$

SO...

$K^\mu = q F^{\mu\nu} \eta_\nu$

or equiv. $K^\mu = q F^\mu{}_\nu \eta^\nu$

NOT 3504

!! ADVANCED !! what about Lagrangian?

$$L = \frac{-mc^2}{\gamma} + Q\vec{v}\cdot\vec{A} - Q\phi \qquad \text{will do (good "guess")}$$

x comp:

$$\frac{\partial L}{\partial v_x} = \gamma m v_x + QA_x = P_x \qquad \text{"canonical momentum"}$$

$$\frac{d}{dt}\frac{\partial L}{\partial v_x} = \frac{d}{dt}(\gamma m v_x) + Q\left(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}v_x + \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_x}{\partial z}v_z\right)$$

$$\frac{\partial L}{\partial x} = -Q\frac{\partial \phi}{\partial x} + Q\left(v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right) \qquad \text{but: } \frac{d}{dt}\frac{\partial L}{\partial v_x} = \frac{\partial L}{\partial x}$$

$$\text{so: } \frac{d}{dt}(\gamma m v_x) = -Q\underbrace{\left(\frac{\partial \phi}{\partial x} + \frac{\partial A_x}{\partial t}\right)}_{E_x} + Q\left(v_y\underbrace{\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]}_{B_z} + v_z\underbrace{\left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right]}_{B_y}\right)$$

$$= QE_x + Q(\vec{v}\times\vec{B})_x = \frac{dp_x}{dt} = F_x \checkmark$$

$$H = \vec{P}\cdot\vec{v} - L \qquad = \frac{\vec{P}\cdot\vec{d}c}{\sqrt{m^2c^2+d^2}} + \frac{m^2c^3}{\sqrt{m^2c^2+d^2}} - \frac{Q\vec{A}\cdot\vec{d}c}{\sqrt{m^2c^2+d^2}} + Q\phi$$

$$\gamma m\vec{v} = \underbrace{\vec{P} - Q\vec{A}}_{\vec{d}}$$

$$\frac{m^2v^2}{1-\frac{v^2}{c^2}} = d^2 \qquad\qquad = \frac{\left[(\vec{P}-Q\vec{A})^2 + m^2c^2\right]c}{\sqrt{(\vec{P}-Q\vec{A})^2+m^2c^2}} + Q\phi$$

$$m^2v^2 = d^2 - \frac{d^2v^2}{c^2} \qquad\qquad = \sqrt{(\vec{P}-Q\vec{A})c^2 + m^2c^4} + Q\phi$$

$$v^2\left(m^2 + \frac{d^2}{c^2}\right) = d^2$$

$$\vec{v} = \frac{\vec{d}c}{\sqrt{m^2c^2+d^2}} \qquad \text{recall: } E = \sqrt{p^2c^2 + m^2c^4}$$

$$\frac{v^2}{c^2} = \frac{d^2}{m^2c^2+d^2} \qquad \text{so for E+M } \vec{p} \to \vec{P} - Q\vec{A} \text{ + add } Q\phi$$

$$1 - \frac{v^2}{c^2} = \frac{m^2c^2 + \cancel{d^2} - \cancel{d^2}}{m^2c^2+d^2}$$

$$\gamma = \frac{\sqrt{m^2c^2+d^2}}{mc}$$