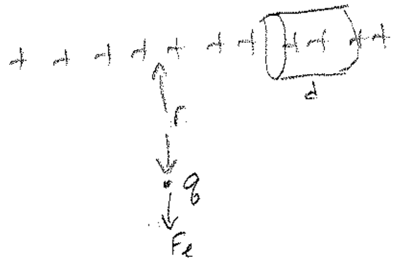
Consider:



$$\oint E \cdot dA = \frac{q_{enc}}{\epsilon_0}$$

$$E\, 2\pi r d = \frac{\lambda d}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi r \epsilon_0}$$

consider $F_\perp$ @ rest $F_E = qE$

$$= \frac{q\lambda}{2\pi\epsilon_0 r}$$

from frame moving right at speed v

$$\bar{F}_E = \frac{q\gamma\lambda}{2\pi\epsilon_0 r} \quad \left(\text{since length is contracted } \lambda \to \gamma\lambda\right)$$

but in this case (from last page) $\bar{F}_\perp = \frac{1}{\gamma} F_\perp$

so something is missing in moving frame

$$\bar{F} = \frac{q\gamma\lambda}{2\pi\epsilon_0 r} + \bar{F}_{new} = \frac{1}{\gamma}F_\perp = \frac{1}{\gamma}\left(\frac{q\lambda}{2\pi\epsilon_0 r}\right)$$

$$\bar{F}_{new} = \frac{1}{\gamma}\frac{q\lambda}{2\pi\epsilon_0 r} - \frac{q\lambda\gamma}{2\pi\epsilon_0 r} = \frac{q\gamma\lambda}{2\pi\epsilon_0 r}\left(\frac{1}{\gamma^2} - 1\right)$$

but $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$\beta^2 = 1 - \frac{1}{\gamma^2}$

$$= -\frac{q\gamma\lambda}{2\pi\epsilon_0 r}\frac{v^2}{c^2} = -\frac{q v \mu_0 \cancel{\epsilon_0} \bar{\lambda} v}{2\pi \cancel{\epsilon_0} r}$$

$\bar{\lambda} = \gamma\lambda$

(recall $c^2 = \frac{1}{\mu_0\epsilon_0}$)

$$= -qv\left(\frac{\mu_0 \bar{I}}{2\pi r}\right) = -q v \bar{B}$$

[recall $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$

$B = \frac{\mu_0 I_{enc}}{2\pi r}$]

so @ rest only F_E

but @ v get both $\bar{F}_E + \bar{F}_B$

so, different frames will see mixing of $\vec{E}$ + $\vec{B}$ fields

$\Rightarrow$ B^μ + E^μ can $\boxed{\text{not}}$ be simply 4-vectors

since $\bar{B}^\mu = \Lambda^\mu{}_\nu B^\nu$ could not mix them...

(A) recap: 4-vectors we've seen

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad dx^\mu = \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix}$$

$$\eta^\mu = \frac{dx^\mu}{d\tau} = \begin{pmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix} = \begin{pmatrix} \gamma c \\ \frac{dx}{d\tau} \\ \frac{dy}{d\tau} \\ \frac{dz}{d\tau} \end{pmatrix} \qquad p^\mu = m\eta^\mu = \begin{pmatrix} E/c \\ p^x \\ p^y \\ p^z \end{pmatrix} \qquad \left(\begin{array}{l} \text{recall:} \\ E = \gamma m c^2 \end{array}\right)$$

(B) Lorentz scalars we've seen (same in all frames)

$$x^\mu x_\mu = -c^2t^2 + x^2 + y^2 + z^2 = -c^2t^2 + \vec{x}^2$$

$$dx^\mu dx_\mu = -c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2dt^2 + d\vec{x}^2 = -c^2 d\tau^2$$

$\Rightarrow d\tau$ is L.I.

$$\eta^\mu \eta_\mu = -c^2\gamma^2 + \vec{\eta}^2 = -c^2$$

$$p^\mu p_\mu = -\frac{E^2}{c^2} + \vec{p}^2 = -m^2c^2 \Rightarrow E^2 = \vec{p}^2c^2 + m^2c^4$$

(C)

Contra-variant to co: $x_\mu = g_{\mu\nu} x^\nu$

Forces: $\vec{F} = \frac{d\vec{p}}{dt}$

$K^\mu = \frac{dp^\mu}{d\tau}$ = "minkowsky Force" (4-vector)

(D) Switching Frames:

$$\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \bar{x}_\mu = \Lambda_\mu{}^\nu x_\nu$$

where x^ν is a 4-vector

Λ options: rotate about x: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{pmatrix}$ y: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & -s \\ 0 & 0 & 1 & 0 \\ 0 & s & 0 & c \end{pmatrix}$ z: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\left(\begin{array}{l} c = \cos\theta \\ s = \sin\theta \end{array}\right)$

boost along x: $\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ y: $\begin{pmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ z: $\begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$

(E) Special Cases: (ie dt instead of $d\tau$)

1) normal forces $= \frac{d\vec{p}}{dt}$

ie, for boost along x

$$\bar{F}_y = \frac{F_y}{\gamma\left(1 - \beta \frac{u_x}{c}\right)}$$

$$\bar{F}_z = \frac{F_z}{\gamma\left(1 - \beta \frac{u_x}{c}\right)}$$

$$\bar{F}_x = \frac{F_x - \beta\left(\vec{u}\cdot\vec{F}\right)/c}{\left(1 - \beta \frac{u_x}{c}\right)}$$

2) normal velocity $= \frac{d\vec{x}}{dt}$

ie boost along x

$$\bar{u}_x = \frac{u_x - v}{\left(1 - \beta \frac{u_x}{c}\right)}$$

$$\bar{u}_y = \frac{u_y}{\gamma\left(1 - \beta \frac{u_x}{c}\right)}$$

$$\bar{u}_z = \frac{u_z}{\gamma\left(1 - \beta \frac{u_x}{c}\right)}$$

∴ Consider Lorentz force: $\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{u} \times \vec{B})$ eqn of space comp. (3-vectors)

$$\frac{d\vec{p}}{dt}\frac{dt}{d\tau} = q\left(\frac{dt}{d\tau}\vec{E} + \frac{dt}{d\tau}\vec{u} \times \vec{B}\right)$$

$$\frac{d\vec{p}}{d\tau} = q\left(\gamma \vec{E} + \vec{\eta} \times \vec{B}\right) = q\left(\frac{\eta^0}{c}\vec{E} + \vec{\eta} \times \vec{B}\right)$$

$\eta^\mu = \gamma u^\mu$

$\eta^0 = \frac{dct}{d\tau} = \gamma c$

$\frac{dt}{d\tau} = \gamma = \frac{\eta^0}{c}$

what about the "0" component?

$$\frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau} \qquad (p^0 = E/c)$$

$$= \frac{1}{c}\frac{d\vec{p}}{d\tau}\cdot\vec{u} \qquad \left(\frac{d\vec{p}}{dt}\cdot\vec{u} = \frac{dE}{dt} \text{ from w-e theorem proof}\right)$$

$$= \frac{1}{c} q\left(\gamma \vec{E} + \vec{\eta} \times \vec{B}\right)\cdot\frac{\vec{\eta}}{\gamma}$$

$$= q\,\vec{\eta}\cdot\frac{\vec{E}}{c}$$

(note: $\vec{A}\cdot(\vec{A} \times \vec{B}) \equiv 0$
so $\vec{B}$ field does not
do any work, as expected)

so let's put components together:

note $\vec{\eta} \times \vec{B} = \begin{pmatrix} e_1 & e_2 & e_3 \\ \eta_1 & \eta_2 & \eta_3 \\ B_1 & B_2 & B_3 \end{pmatrix}$

$$\begin{pmatrix} \frac{dp^0}{d\tau} \\ \frac{dp^1}{d\tau} \\ \frac{dp^2}{d\tau} \\ \frac{dp^3}{d\tau} \end{pmatrix} = q \underbrace{\begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}}_{\equiv F^{\mu\nu}} \begin{pmatrix} -\eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} \leftarrow \text{covariant form}$$

notice $F^{\mu\nu} = -F^{\nu\mu}$
"totally anti-symmetric"
⇒ only 6 free parameters

$$K^\mu = q\, F^{\mu\nu}\,\eta_\nu$$

how does this transform?

$$\Lambda^\alpha{}_\mu K^\mu = q\, \underbrace{\Lambda^\alpha{}_\mu F^{\mu\nu} \Lambda^t{}_\nu{}^\beta}_{\bar{F}^{\alpha\beta}}\, \overbrace{\underbrace{\Lambda_\beta{}^\gamma\,\eta_\gamma}_{\bar{\eta}_\beta}}^{\delta^\gamma_\nu}$$

$$\bar{K}^\alpha = q\;\; \bar{F}^{\alpha\beta}\;\; \bar{\eta}_\beta \quad \text{so: } \bar{F}^{\alpha\beta} = \Lambda^\alpha{}_\mu F^{\mu\nu} \Lambda^t{}_\nu{}^\beta$$

$$\bar{K}^\alpha = q\, F^{\alpha\beta}\, \bar{\eta}_\beta$$

So, try boost in x-direction:

$$\text{lamda} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; = \Lambda^t$$

$$f = \begin{pmatrix} 0 & Ex/c & Ey/c & Ez/c \\ -Ex/c & 0 & Bz & -By \\ -Ey/c & -Bz & 0 & Bx \\ -Ez/c & By & -Bx & 0 \end{pmatrix};$$

transform = lamda . f . lamda^t; = lamda . f . Transpose[lamda]

MatrixForm[%]

$$\frac{E_x}{c}\gamma^2(1-\beta^2) = \frac{E_x}{c}$$

$$\begin{pmatrix} 0 & \frac{Ex\gamma^2}{c} - \frac{Ex\beta^2\gamma^2}{c} & \frac{Ey\gamma}{c} - Bz\beta\gamma & \frac{Ez\gamma}{c} + By\beta\gamma \\ -\frac{Ex\gamma^2}{c} + \frac{Ex\beta^2\gamma^2}{c} & 0 & Bz\gamma - \frac{Ey\beta\gamma}{c} & -By\gamma - \frac{Ez\beta\gamma}{c} \\ -\frac{Ey\gamma}{c} + Bz\beta\gamma & -Bz\gamma + \frac{Ey\beta\gamma}{c} & 0 & Bx \\ -\frac{Ez\gamma}{c} - By\beta\gamma & By\gamma + \frac{Ez\beta\gamma}{c} & -Bx & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{\gamma(E_y - vB_z)}{c} & \frac{\gamma(E_z + vB_y)}{c} \\ & 0 & \gamma\left(B_z - \frac{vE_y}{c^2}\right) & -\gamma\left(B_y + \frac{vE_z}{c^2}\right) \\ & & 0 & B_x \\ & & & 0 \end{pmatrix}$$

$$\frac{\bar{E}_z}{c}$$

ie:

$$\bar{E}_x = E_x$$
$$\bar{E}_y = \gamma(E_y - vB_z)$$
$$\bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x$$
$$\bar{B}_y = \gamma\left(B_y + \frac{vE_z}{c^2}\right)$$
$$\bar{B}_z = \gamma\left(B_z - \frac{vE_y}{c^2}\right)$$

as in book

note:

$$\vec{E}' = E_\parallel + \gamma(E_\perp + \vec{v}\times\vec{B})$$
$$\vec{B}' = B_\parallel + \gamma(B_\perp - \vec{v}\times\vec{E}/c^2)$$

notice

$E \to Bc \quad B \to -\frac{E}{c}$ gives same transformations

ie: $\vec{E}' = E_{\parallel} + \gamma\left(E_{\perp} + \vec{v} \times \vec{B}\right)$

$$\Rightarrow \quad \vec{B}'c = B_{\parallel}c + \gamma\left(B_{\perp}c + \vec{v} \times \left(-\frac{\vec{E}}{c}\right)\right)$$

$$\vec{B}' = B_{\parallel} + \gamma\left(B_{\perp} - \vec{v} \times \vec{E}/c^2\right)$$

so if $F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$

then its "dual" $G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$

this is related to the arbitrary assignment of having electric charges, rather than magnetic ones.